Stage 5: Mathematics STEM Advanced Pathway

Sample concentrated study: Factorise and expand

The Concentrated Studies focus on skills from within one Strand.

**Duration:** 5 weeks

*Note to teachers: this Content is suggested to sit between Quick Sketch and Detailed Sketch and in those programs students will engage with STEM applications of the algebraic processes which are the focus of this concentrated study.*

**Outcomes**

A student:

- simplifies algebraic fractions, and expands and factorises quadratic expressions (MA5.2-6NA)
- selects and applies appropriate algebraic techniques to operate with algebraic expressions (MA5.3-5NA)

**Content**

Students:

- apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate (ACMNA213)
- expand algebraic expressions, including those involving terms with indices and/or negative coefficients, e.g. $-3x^2(5x^2 + 2x^4y)$
- expand algebraic expressions by removing grouping symbols and collecting like terms where applicable, e.g. expand and simplify $2y(y - 5) + 4(y - 5)$, $4x(3x + 2) - (x - 1)$
- expand binomial products and factorise monic quadratic expressions using a variety of strategies (ACMNA233)
- expand binomial products by finding the areas of rectangles, e.g.

\[
\begin{array}{c|c|c|c}
& x & 8 \\
\hline
x^2 & 8x \\
3x & 24 \\
\end{array}
\]

hence $(x + 8)(x + 3) = x^2 + 3x + 8x + 24 = x^2 + 11x + 24$
- use algebraic methods to expand binomial products, e.g. $(x + 2)(x - 3)$, $(4a - 1)(3a + 2)$
- expand binomial products using a variety of strategies (ACMNA233)
- recognise and apply the special product, $(a - b)(a + b) = a^2 - b^2$
- recognise and name appropriate expressions as the 'difference of two squares' (Communicating)
- recognise and apply the special products, $(a + b)^2 = a^2 + 2ab + b^2$  
  $(a - b)^2 = a^2 - 2ab + b^2$
- recognise and name appropriate expressions as 'perfect squares' (Communicating)
- simplify a variety of expressions involving binomial products, e.g. $(3x + 1)(2 - x) + 2x + 4$, $(x - y)^2 - (x + y)^2$
- factorise algebraic expressions by taking out a common algebraic factor (ACMNA230)
- factorise algebraic expressions, including those involving indices, by determining common factors, e.g. factorise $3x^2 - 6x$, $14ab + 12a^2$, $21xy - 3x + 9x^2$, $15p^2q^3 - 12pq^4$
- recognise that expressions such as \(24x^2y + 16y^2 = 4xy(6x + 4y)\) may represent 'partial factorisation' and that further factorisation is necessary to 'factorise fully' (Reasoning)
- factorise monic quadratic trinomial expressions, eg \(x^2 + 5x + 6, x^2 + 2x - 8\)
- factorise monic and non-monic quadratic expressions (ACMNA269)
- factorise algebraic expressions, including those involving:
  - common factors
  - a difference of two squares
  - grouping in pairs for four-term expressions
  - perfect squares
  - quadratic trinomials (monic and non-monic)
- use a variety of strategies to factorise algebraic expressions, eg \(3d^3 - 3d, 2a^2 + 12a + 18, 4x^2 - 20x + 25, t^2 - 3t + st - 3s, 2a^2b - 6ab - 3a + 9\)
- simplify a variety of expressions involving binomial products, eg \((3x + 1)(2 - x) + 2x + 4, (x - y)^2 - (x + y)^2\)
- explain why a particular algebraic expansion or factorisation is incorrect, eg 'Why is the factorisation \(x^2 - 6x - 8 = (x - 4)(x - 2)\) incorrect?' (Communicating, Reasoning)

**Common misconceptions**

Students may:
- find the instruction 'expand' confusing when it results in a 'smaller' expression than was given in the question, eg expand \((x + 3)(x - 3)\) gives \(x^2 - 9\)
- be frustrated by algebraic work that does not end in a 'solution'
- be unsure about whether they have 'finished' a question when it does not end in a solution
- have difficulty accepting that factorised and expanded forms of expressions can look very different but be equivalent
- ask questions such as – 'But where did the \(x^2\) go?'
- not yet be able to distinguish whether ‘-' is indicating an operation or a direction and hence make errors when identifying common factors involving negative numbers and expansions involving negative values
- may not yet be certain of index notation and interpret \(x^2\) as \(2x\)
- have been practising substitution skills without understanding the process, hence do not trust substitution as a method of simplifying a complex expression
- still be developing 'multiplicative thinking' and hence slow to recognise common factors or apply the distributive law
- perceive the choice of methods as a problem, preferring there to be 'one best method'.

**Link to calculus-based courses**

Factorisation and expansion become essential tools for the algebraic manipulation required in later studies of mathematics.

**Teaching and Learning**

**STEM platform – Information**

Information is not always presented to us in the most useful form. Often the key characteristic or feature that we are looking for is ‘hidden’ inside the information. This can also be the case for objects - the part you want is hidden within.
Re-organising, sorting and filtering information and opening up or taking apart an object can reveal the thing we are looking for.

Expanding and factorising are algebraic methods of re-organising or opening up an expression. Just as all the pieces of an object still exist after it is taken apart, so do all the pieces of an expression exist after it has been factorised or expanded. However, in both cases, the appearance will have changed.

**Pre-test** – ‘Factorisation and Expansion’: [https://www.tes.com/teaching-resource/pre-test-factorisation-and-expansion-11423638](https://www.tes.com/teaching-resource/pre-test-factorisation-and-expansion-11423638). If indicated, a review of Stage 4 expansion and factorisation can be achieved through a series of worksheets from 'STEM Learning': [https://www.stem.org.uk/elibrary/resource/25849](https://www.stem.org.uk/elibrary/resource/25849). This site requires registration - which is free. Detailed lesson plans and teaching notes are also available.) Or, school maths texts may provide equivalent resources.

Encourage students to memorise their times tables with a range of games such as ‘Factors and Multiples’: [http://nrich.maths.org/8146](http://nrich.maths.org/8146).

**Introduction activities**


Students play against a computer, or draw dots to create a game grid to play against one another.

Reinforcing that square numbers also have a physical length and width will be useful for later applications.


This puzzle is a good case for the usefulness and need for algebraic evidence in problem-solving, particularly when calculating area. Our eyes can deceive us and our minds can struggle to make sense of what we are seeing – the algebra will always be reliable.


Increase student speed and confidence in recognising and distinguishing between factors and multiples with this game. Grids can be printed if school resources do not allow all students to engage with the interactive.


Revise and develop student ability to recognise like terms

**Foundation understanding: Equivalence of expanded and factorised forms**

Students copy, complete and then extend this table with further examples.

<table>
<thead>
<tr>
<th>Simplified</th>
<th>Written as factors</th>
<th>Alternative factors</th>
<th>Fully factorised</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$6 \times 2$</td>
<td>$4 \times 3 \times 1$</td>
<td>$3 \times 2 \times 2$</td>
</tr>
<tr>
<td>$-16$</td>
<td>$16 \times -1$</td>
<td>$-4 \times 4$</td>
<td></td>
</tr>
<tr>
<td>$6a$</td>
<td>$6 \times a$</td>
<td>$a \times 2 \times 3$</td>
<td></td>
</tr>
<tr>
<td>$3a^2$</td>
<td>$3 \times a^2$</td>
<td>$a \times 3 \times a$</td>
<td></td>
</tr>
</tbody>
</table>
### Simplified | Written as factors | Alternative factors | Fully factorised
--- | --- | --- | ---
$9a \times b$ | $3 \times 3ab$ | $x^2 \times 4$ | $m \times -m \times 2 \times 5$
$18m - 12$ | $2(9m - 6)$ | $a(3 + b)$ | $4 \times p(p + 6)$

**Link to learning**

Students who have recently completed ‘Quick Sketch’ will know that different presentations of quadratics are useful for identification of different parts of a graph.

- Fully expanded form – reveals $y$-intercept
- Fully factorised form – reveals $x$-intercepts where they exist
- Partially factorised form – reveals turning point or vertex

Being able to factorise, partially factorise and expand a quadratic empowers students to create complete graphs, and hence answer application questions completely.

Demonstrate the power of a graph to store information and provide answers to problems.

Video: 'Why maths is important': [https://www.youtube.com/watch?v=gDkd0Vaxf-c&feature=youtu.be](https://www.youtube.com/watch?v=gDkd0Vaxf-c&feature=youtu.be)

Discuss:
- What shape is this slide? What shape is the man’s flight path?
- Do you think this video is real or a hoax? Give reasons.
- Whether you think the video real or not, is it possible?
- Scale diagram – equation for take-off and landing
- Test with equation and graph rather than with actual person

**STEM application: design, engineering and mathematics**

For students:

- Imagine you are designing a ride for a theme park – you need it to be exciting, but not terrifying – and definitely not life-endangering.
- A parabola, and in particular, its quadratic equation, might help you decide on the perfect balance between fun and fear

**Consolidation for skill development and guided practice**

The teacher explicitly teaches:

- algebraic methods to expand and simplify algebraic expressions
- the explanation with graphics, available at ‘Algebra – Expanding’:
**Mathematics STEM Advanced Pathway**

**Sample concentrated study unit:** Factorise and Expand

**NSW Education Standards Authority**


**Differentiation:**


- **Structured** - Expand and Simplify Single Brackets
- **Structured** - Expand double brackets – skill blocks
- **Extension** - Expand double brackets – stretch blocks

The teacher explicitly teaches:

- expanding binomial products by finding the areas of rectangles

Students watch the video: ‘Expanding a Binomial x a Binomial using the rectangular method’: https://www.youtube.com/watch?v=vY0sp_oJHo&feature=youtu.be – Note to teachers: this is too ‘dry’ to present to a class but is useful for teachers who have not seen this method

Students complete the printable learning activity: ‘Describing Area’: https://www.mathspad.co.uk/teach/linkedDocuments/expressions/describeAreaBrackets.pdf

**Differentiation:**

- **Structured** - working in pairs complete the worksheet: ‘Expanding Double Brackets’: https://www.tes.com/teaching-resource/expanding-double-brackets-6343009 which breaks down the method, and play the game: ‘Expand and simplify (1) – Connect 4’: http://justmaths.co.uk/Worksheets/Algebra/Expand%20and%20simplify%201%20-%20Connect%204.pdf
- **Extension** - working in pairs students play the game: ‘Expand and simplify (2) – Connect 4’: http://justmaths.co.uk/Worksheets/Algebra/Expand%20and%20simplify%202%20-%20Connect%204.pdf
- **Extension** - Consider the problem ‘Look before you Leap’: http://nrich.maths.org/762 – an expansion problem – area calculation given perimeter in three algebraic terms. This ‘working beyond’ problem might be useful for students who understand binomial expansion, but need something more challenging to help them develop resilience.

The teacher explicitly teaches:

- how to recognise common factors and hence factorise expressions
- a revision of index notation, eg $a^5 = a \times a \times a \times a \times a = a^4 \times a = a^2 \times a^3$ etc.
- that when a common factor cannot be found for all terms, students should look for common factors for pairs of terms
- how to check for further common factors before deciding a question is complete

**Differentiation:**

- **Structured** - school-based or online worksheets such as: ‘Simple factorisation’: https://portal.uea.ac.uk/documents/6207125/8194860/steps+into+algebra+simple+factorisation+worksheet.pdf/e794513d-d44d-409a-9f86-254fb7f87cb
- **Extension** - play the game: ‘Number Cop’: http://assets.varietytutors.com/assets/vt-hotmath-legacy/hotmath_help/games/numbercop/numbercop_hotmath.swf, which can be set to different degrees of difficulty
The teacher explicitly teaches:

- how to factorise quadratic trinomials (monic)

  *Note to teachers: Teachers differ in their preferred methods and students can bring in ‘different’ methods from tutors/parents/friends. It is important that teachers accept all valid methods and assist students in recognising the commonalities between these.*

- that the first step is still looking for common factors

Students:


- read the resource ‘Factorising trinomials’: [http://www.bbc.co.uk/education/guides/zmvrd2p/revision/3](http://www.bbc.co.uk/education/guides/zmvrd2p/revision/3), where the method explained for students (note to teachers: unfortunately this page uses a font resulting in "x" as the multiplication symbol and this needs to be clarified for students.)

**Differentiation:**


The teacher explicitly teaches:

- how to factorise quadratic trinomials (non-monic), using the following resources:
  - ‘Factorising trinomials – extension’: [http://www.bbc.co.uk/education/guides/zmvrd2p/revision/4](http://www.bbc.co.uk/education/guides/zmvrd2p/revision/4) shows the ‘split the middle term’ method explained for students (note to teachers: unfortunately this page uses a font resulting in “x” as the multiplication symbol and this needs to be clarified for students.)

**Differentiation:**

- *Structured* - students should be given problems that become monic once a common factor is ‘removed’


**Special products as effective ‘short cuts’ for expansion and factorisation**

Students discuss:

- Has it been necessary to use some ‘trial and error’ in the factorisation methods used so far?
- Has it sometimes felt like there wasn’t a clear set of rules?
- Was it a little too easy to make an error with negative values when expanding?

The teacher leads a discussion on the following:

Mathematics is full of ‘shortcuts’, in fact, mathematicians might be described as ‘masters of the shortcut’. Every formula is the result of someone realising that there was a pattern to the answers they found for certain types of questions. Even the multiplication table is just a shortcut to having to add large ‘groups of groups’.
STEM stimulus
Students brainstorm technology they use daily as a 'shortcut', i.e. where technology is saving them time and/or effort and where it is reducing the chance of them making mistakes.

In quadratic equations there are two sets of particularly useful rules that cut our factorising and expanding time in half (at least), and help prevent unnecessary mistakes.

**Difference of two squares** – recognise, name and apply the special product, 
\[(a - b)(a + b) = a^2 - b^2\]

**Perfect squares** – recognise, name and apply the special products, 
\[
\begin{align*}
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a - b)^2 &= a^2 - 2ab + b^2
\end{align*}
\]

Activity
- Students complete the worksheet: ‘Recognise Special Products’: [https://www.tes.com/teaching-resource/recognise-special-products-11423641](https://www.tes.com/teaching-resource/recognise-special-products-11423641) - to be completed as a class with teacher guidance
- Students play the game: ‘Square root concentration’: [http://www.math-play.com/Square-Root-Concentration-Game/square-root-concentration-game.html](http://www.math-play.com/Square-Root-Concentration-Game/square-root-concentration-game.html) to reinforce that these are values students should become quick to recognise

Guided practice
School-based and online worksheets could be used as resources.

Differentiation:
- **Structured** - Use substitution to verify answers:
- **Structured** - Have students select a variety of questions and answers that they have completed during this unit and substitute a value for the pronumeral into the expanded and factorised forms of the expression to test/prove their equivalence
- **Extension** - Using substitution to reduce the complexity of an expression for factorisation or expansion:
- **Extension** - Watch the video: ‘Factoring Complicated Expressions – Concept’: [https://www.brightstorm.com/math/algebra-2/factoring/factoring-complicated-expressions/](https://www.brightstorm.com/math/algebra-2/factoring/factoring-complicated-expressions/) which has a set of worked problems. After viewing the explanation video, students can progress through the example videos, pausing each to attempt the answer to each problem before playing the solution.

*Note to teachers: Students will have ongoing opportunity to practise these skills in the context of rearranging equations for graphing purposes in the subsequent unit, Detailed Sketch.*

Resources
**Additional resources for teachers URLs:**
Teaching and Learning URLs for linked items

- ‘STEM Learning’ (STEM Academy): https://www.stem.org.uk/elibrary/resource/25849
- ‘Factors and Multiples’ (NRICH): http://nrich.maths.org/8146
- Video: ‘Why maths is important’: https://www.youtube.com/watch?v=gDkd0Vaxf-c&feature=youtu.be
- Video: ‘Expanding a Binomial x a Binomial using the rectangular method’: https://www.youtube.com/watch?v=vY0sp_oJlHo&feature=youtu.be
- ‘Describing Area’ (Maths Pad): https://www.mathspad.co.uk/teach/linkedDocuments/expressions/describeAreaBrackets.pdf
- ‘Expand and simplify (1) – Connect 4’ (Just Maths): http://justmaths.co.uk/Worksheets/Algebra/Expand%20and%20simplify%201%20-%20Connect%204.pdf
- ‘Expand and simplify (2) – Connect 4’ (Just Maths): http://justmaths.co.uk/Worksheets/Algebra/Expand%20and%20simplify%202%20-%20Connect%204.pdf
- ‘Look before you Leap’: http://nrich.maths.org/762
- ‘Simple factorisation’: https://portal.uea.ac.uk/documents/6207125/8194860/steps+into+algebra+simple+factorisation+worksheet.pdf/e794513d-d44d-409a-9f86-2548b7f87cb
- Video and worked examples: ‘Factoring by Grouping’ (Shmoop): http://www.shmoop.com/polynomials/grouping.html
- ‘Number Cop’: http://assets.varsitytutors.com/assets/vt-hotmath-legacy/hotmath_help/games/numbercop/numbercop_hotmath.swf
- ‘Factorising trinomials’ (BBC): http://www.bbc.co.uk/education/guides/zmvr2p/revision/3
- ‘Factorising trinomials – extension’: http://www.bbc.co.uk/education/guides/zmvr2p/revision/4
- ‘Year 10 Interactive Maths – Cross-multiplication Method’: 